MAAT SEMINAR, CRMEF RABAT

Self Homotopy Equivalences

Theriault's Conjecture

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Theriault Open Question

What one may say about the finitude of the homotopical nilpotency of the monoid of self homotopy equivalence whenever the cocategory of its classifying space is ?

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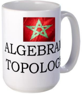
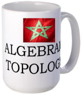
Ben El Krafi-M. (2015)

In rational context : Yes and More



Content



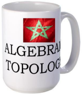
Recalling

X topological space

Self Homotopy Equivalences

$\mathrm{aut}(X)$ denotes the monoid of its self homotopy equivalences, that are maps $f : X \to X$ which admits a homotopy inverse.

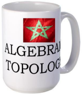
$\mathrm{Baut}(X)$ denotes the associated classifying space.



Recalling

Nilpotency

For a topological monoid G, the homotopical nilpotency of G, denoted $\mathrm{Hnil}(G)$, is then the least integer n such that $n+1$-th commutator c_{n+1} is nullhomotopic, where iterated commutators $c_n : G^n \to G$ are inductively defined, using the homotopy inverse as follows : c_1 is the identity, $c_2(a, b) := aba^{-1}b^{-1}$ and $c_n := c_2 \circ (c_{n-1}, c_1)$.



Recalling

Quillen model

That is a differential graded Lie algebra, $(\mathbb{L}_W, \partial)$, where W is a graded vector space, with $\partial(\mathbb{L}_W) \subset \mathbb{L}_W^{\geq 2}$

D. Quillen (1967)

Any simply connected and rational CW-complex of finite type, X, admits a minimal Quillen model $(\mathbb{L}_W, \partial)$, unique up to isomorphism, which encodes the rational homotopy type as follows :

$$\begin{aligned} H_*(\mathbb{L}_W, d) &\cong \pi_{*+1}(X) \otimes \mathbb{Q} \\ W &\cong \tilde{H}_{*+1}(X; \mathbb{Q}) \end{aligned}.$$

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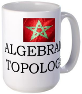
Recalling

Cocategory

the rational cocategory of X, denoted $\mathrm{cocat}_{\mathbb{Q}}(X)$, is defined to be the smallest integer (or infinite) such that the projection $(\mathbb{L}_W, \partial) \to (\mathbb{L}_W/\mathbb{L}_W^{\geq n+1}, \partial)$ admits a retract.



Recalling

Formal-Coformal Spaces

Formal

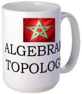
$$\partial(\mathbb{L}_W) \subset \mathbb{L}_W^2.$$

The differential is purely quadratic

CoFormal

$$\mathbb{L}_W \cong \pi_{*+1}(X) \otimes \mathbb{Q}.$$

The homotopy can formally obtained from the homology



Recalling

Formal-Coformal Spaces

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$$\partial(\mathbb{L}_W) \subset \mathbb{L}^2_W.$$

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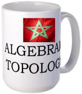
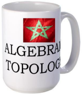


Our results

Theorem 1

Let X be a simply connected CW-complex of finite type. If $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}(X))$ is finite, then $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X))$ is also. Moreover, we have

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X)) \leqslant \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}(X)).$$



Our results

Proposition 1.1

Let X be a simply connected CW-complex of finite type, such that $\mathrm{Baut}(X)$ is coformal. If $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}(X))$ is finite, then $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X))$ is also. Moreover, we have

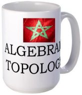
$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X)) = \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}(X)).$$

Our results

Proposition 1.2

Let X be a simply connected CW-complex of finite type, such that $\mathrm{Baut}(X)$ is formal. If $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}(X))$ is finite, then $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X))$ is also. Moreover, we have

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X)) \leqslant 2.$$



Applications

- the inequality $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X)) > 2$ is an obstruction for the formality of $\mathrm{Baut}(X)$, when $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}(X))$ is finite ;
- Unfortunately, in general $\mathrm{Baut}(X)$ is not nilpotent and so it is not possible (or easy, anyway) to talk about its formality.

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Exit issue

- The path components of a well-pointed grouplike space are all of the same homotopy type ;
- We focus on $\mathrm{Baut}_1(X)$, where $\mathrm{aut}_1(X)$ denotes the path component of the identity map ;
- The formality of $\mathrm{Baut}_1(X)$ is well studied by Smith ;
- The respective Sullivan and Quillen minimal models of both $\mathrm{aut}_1(X)$ and $\mathrm{Baut}_1(X)$ are well and deeply described in terms of derivations by Felix, Buijs and Smith ;
- $\mathrm{aut}_1(X)$ and its classifying space play a crucial role in topology and geometry (Stasheff's classification for fibration over a given fiber, fake Lie groups, the homotopy of the diffeomorphisms on smooth manifold, ...).

Our results

Theorem 2

Let X be a simply connected CW-complex of finite type. If $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ is finite, then $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X))$ is also. Moreover, we have

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) \leqslant \mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X)).$$

Our results

Proposition 2.1

Let X be a simply connected CW-complex of finite type, such that $\mathrm{Baut}_1(X)$ is coformal. If $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ is finite, then $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}(X))$ is also. Moreover, we have

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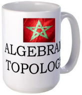


Our results

Proposition 2.2

Let X be a simply connected CW-complex of finite type, such that $\mathrm{Baut}_1(X)$ is formal. If $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ is finite, then $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X))$ is also. Moreover, we have

$$\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) \leqslant 2.$$

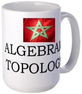
Intersected results

Open Problem 1

When $\mathrm{Baut}_1(X)$ is a rational H-space ?

Intersected results

- H-spaces are formal ;
- the inequality $\mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) > 2$ is an obstruction for the formality of $\mathrm{Baut}_1(X)$, when $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ is finite ;
- $\mathrm{Baut}_1(G)$ is rarely H-space when G is a topological group ;



Intersected results

Intersected result 1

If G is a connected topological group such that $\mathrm{Baut}_1(G)$ is a H-space of finite cocategory, then $\mathrm{Hnil}(G) \leq 2$.

Intersected results

Open Problem 2 : Halperin Conjecture

The rational Serre spectral sequence collapses at the E_2-term for every fibration of simply connected whose fiber is an elliptic F_0-CW complex.

Intersected results

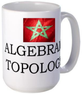
- F_0-space means $H^{\text{odd}}(X; Q) = 0$, in this case $\mathrm{Baut}_1(X)$ is a H-space ;

Intersected results

Intersected result 2

Let X is an elliptic F_0-space such that $\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X))$ is finite. If X verifies the Halperin conjecture, then

$$\mathrm{cocat}_{\mathbb{Q}}(\mathrm{Baut}_1(X)) = \mathrm{Hnil}_{\mathbb{Q}}(\mathrm{aut}_1(X)) = 1.$$

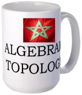


Intersected results

Open Problem 3 : Realizability problem

Any group G is rationally realizable as self homotopy equivalences of a rational 1-connected CW-complex X, i.e.,

$$\mathrm{aut}(X) \cong G$$



Intersected results

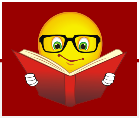
- Yes, when G is finite (Costoya and Viruel)

Intersected results

Intersected result 3

If G is finite group whose a classifying space BG is of finite rational cocategory, then G is of finite rational homotopical nilpotency. Moreover we have $\mathrm{Hnil}_{\mathbb{Q}}(G) \leq \mathrm{cocat}_{\mathbb{Q}}(BG)$.

Main used references



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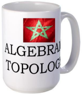
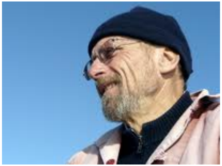


Acknowledgements

J.C. Thomas, Angers, France

Suggesting the subject



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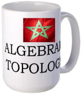
S.B. Smith, Philadephia USA

lectures and the many fruitful conversations



Acknowledgements

MAAT : Moroccan Area of Algebraic Topology

That's all talks

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