On the Hilali conjecture related to the Halprin one

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Idée de départ

Questions de départ:(2012)
1/ dim $Ext_{UL_X}(\mathbb{Q},\mathbb{Q}) \geq$ dim UL_X ?
2/ dim $Ext_{UL_X}(\mathbb{Q},\mathbb{Q}) \geq$ dim L_x?
Eléments de réponse, developpées par J-C-Thomas
$\mathbb{k}$ désigne un corps de caractéristique 0
1/ $C^*(L,0) = (\Lambda V, \partial)$
2/ $V = \sharp sL$
3/ $C_*(L,0) = (\Lambda sL, \partial)$
4/ $C^* = \sharp \circ C_*$
5/
$\partial(sx_1 \Lambda sx_2 \Lambda ... \Lambda sx_k) = \sum_{1 \leq i < j \leq k} (-1)^{|sx_1|+|sx_2|+...|sx_i|} s\left[x_i, x_j\right] \Lambda sx_1 \Lambda ... \Lambda \widehat{sx_i}$
$\Lambda ... \Lambda \ \widehat{sx_j} \Lambda ... \Lambda sx_k$
$\partial(sx) = 0$

Définitions

Definitions

(p.65 [Ta]) Une ldg (L, ∂),1-réduite,à homologie de type fini, est coformelle si l'une des coditions équivalentes suivantes est satisfaite:
i/ Son algèbre de Lie d'homologie $(H_*(L, \partial), 0)$ à même type d'homotopie que (L, ∂).
ii/ (L, ∂) à un modèle de Sullivan à différentielle purement quadratique.

Definitions

Un espace X est coformel si $(L_X = \pi_*(\Omega X) \otimes \mathbb{Q}, 0)$ est un LDG- modèle de X.
$C^*(\pi_*(\Omega X) \otimes \mathbb{Q}, 0)$ est un modèle de Sullivan de X à différentielle quadratique.

Theorem

$$dim(Ext_{UL}(\mathbb{k},\mathbb{k})) \geq dim(L/\left[L;L\right])$$

Proof.

$$\begin{aligned} dim\ (Ext_{UL}(\mathbb{k},\mathbb{k})) &= dim\ H(C^*(L)) \qquad (p.315\ [FHT]) \\ &= dim\ H(C_*(L)) \\ &\geq dim\ H_1(C_*(L)) = dim L/\left[L;L\right] \end{aligned}$$

□

Remarque:
Si L est abélienne alors $Ext_{UL}(\mathbb{k},\mathbb{k}) = \Lambda sL$ donc
$dim(Ext_{UL}(\mathbb{k},\mathbb{k})) \geq dim(L)$

Theorem

Si $L = L^{even}$ *alors*

$$\dim V \leq dim\ H^*(\Lambda V, d)$$

Theorem 1. If $(\Lambda V, d)$ is an elliptic and 1-connected coformal minimal Sullivan model whose associated Quillen model L is concentrated on even degrees (i.e., $L = L^{even}$), then
$\dim V \leq dim\ H^*(\Lambda V, d)$
Remarque:
In fact, the result of C. Deninger and W. Singhof is established for ungraded Lie algebras (graded Lie algebras concentrated in degrees 0), however it can be extended to graded Lie algebras concentrated in even degrees.
Indeed, one may forgot the graduation since the antisymmetry of the Lie bracket and the identity of Jacobi are the same.

Exemples

1/ $\dim L = 3$ et $|e_1| = 4; |e_2| = 6; |e_3| = 10$

$[e_1, e_2] = e_3; [e_1, e_3] = [e_2, e_3] = 0$ (e_1 et e_2 est une base de $L/[L;L]$ et e_3 base de $[L;L]$)

$0 \longrightarrow \Lambda^3 sL \longrightarrow \Lambda^2 sL \longrightarrow sL \longrightarrow 0$

$\partial se_1 = \partial se_2 = \partial(se_1 \Lambda se_2 \Lambda se_3) = \partial(se_1 \Lambda se_3) = \partial(se_2 \Lambda se_3) = 0$

$dim\ H(C_*(L)) = 5$

2/ $\dim L = 3$ et $|e_1| = 1; |e_2| = 2; |e_3| = 4$

$[e_1, e_1] = e_2$ (e_1 et e_3 est une base de $L/[L;L]$ et e_2 base de $[L;L]$)

$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = [e_2, e_2] = [e_3, e_3] = 0.$

$\partial se_1 = \partial se_3 = \partial(se_1 \Lambda se_3) = 0$

$dim\ H(C_*(L)) = 3$

Exemples

3/ dim $L = 3$ et $|e_1| = 3; |e_2| = 5; |e_3| = 8$
$[e_1, e_2] = e_3; [e_1, e_1] = [e_2, e_2] = [e_3, e_3] = 0. [e_1, e_3] = [e_2, e_3] = 0$ (e_1 et e_2 est une base de $L/[L; L]$ et e_3 base de $[L; L]$)
$\partial(se_1^{k_1}) = \partial(se_2^{k_2}) = 0$
$dim\ H(C_*(L)) = \infty$

Exemples

$4/\ \dim L = 4\ |e_1| = 2; |e_2| = 4; |e_3| = 6, |e_4| = 10$
$[e_1, e_2] = e_3; [e_1, e_3] = [e_1, e_4] = 0.$
$[e_2, e_3] = e_4; [e_2, e_4] = 0.$
$[e_3, e_4] = 0.$
$\partial se_1 = \partial se_2 = 0.$
$\partial(se_1 \Lambda se_3) = 0.$
$\partial(se_2 \Lambda se_4) = 0.$
$\partial(se_1 \Lambda se_3 \Lambda se_4) = 0.$
$\partial(se_2 \Lambda se_3 \Lambda se_4) = 0$
$\partial(se_1 \Lambda se_2 \Lambda se_3 \Lambda se_4) = 0.$
$dim\ H(C_*(L)) = 7$

Supposons que $\dim L = n$ et $L = L^{even}$
Remarquons que $L \cong [L, L] \oplus L/[L, L]$
$\{e_1, e_2, ..., e_p, ..., e_q, ..., e_{q+p}\}$une base de $L/[L, L]$ avec $2 \leq p \leq q$
$\{e_{q+p+1}, ..., e_{q+2p}\}$une base de $[L, L]$
telles que:
$[e_i, e_j] = 0$ pour $i \leq j \leq q$ ou $q + 1 \leq i \leq j \leq n = q + 2p$
$[e_i, e_{q+i}] = e_{q+p+i}$ $1 \leq i \leq p$ tous les autre nuls

$$d(se_1^{\epsilon_1} \Lambda\ se_2^{\epsilon_2} \Lambda ... \Lambda\ se_q^{\epsilon_q}) = \sum_{1 \leq i < j \leq q} (-1)^{|se_1^{\epsilon_1}| + |se_2^{\epsilon_2}| + ... |se_i^{\epsilon_i}|} s[e_i, e_j]\ \Lambda\ se_1^{\epsilon_1} \Lambda ... \Lambda\ \widehat{se_i^{\epsilon_i}} \Lambda ... \Lambda\ \widehat{se_j^{\epsilon_j}} \Lambda ... \Lambda\ se_q^{\epsilon_q} = 0$$

$d(se_{q+1}^{\epsilon_{q+1}} \Lambda\ se_{q+2}^{\epsilon_{q+2}} \Lambda ... \Lambda\ se_{q+p}^{\epsilon_{q+p}}) = 0 \qquad (\epsilon_i = 0$ ou $\epsilon_i = 1)$
$d(se_i \Lambda\ se_{q+i} \Lambda\ se_{q+p+i}) = 0$ avec $1 \leq i \leq p$
donc $\dim H_*(C_*(L, 0)) \geq 2^q + 2^p + p - 2 \geq n$

Abstract

In this paper, we focus on Hilali's conjecture that, for any simply connected elliptic CW-complex X, the total sum of the rational Betti numbers is at least as large as the total rank of its rational homotopy. We investigate this conjecture for coformal spaces and suggest some research directions to resolve it completely. Finally, we put up a bridge between the Hilali conjecture and that of Halperin; the toral rank conjecture and use it to establish the latter holds for all manifolds of dimension less than 16 and whose toral rank is equal to 4.

Introduction

Through this paper, X denotes a simply-connected elliptic CW-complex with $(V; d)$ its Sullivan minimal model. For more details on rational minimal models,we refer the reader to the standard book $[FHT01]$. X is said to be elliptic when both $H(X;\mathbb{Q})$ and $\pi_*(X)\otimes\mathbb{Q}$ are of finite dimension. For these spaces, Hilali conjectured in 1990 ($[Hi90]$) that:

Theorem

Hilali Conjecture. If X is a simply-connected elliptic CW-complex, then:

$$\dim \pi_*(X)\otimes\mathbb{Q} \leq \dim H^*(X;\mathbb{Q})$$

Theorem

Hilali Conjecture (Algebraic version). If $(\Lambda V, d)$ is a 1-connected elliptic Sullivan model, then:

$$dimV \leq dimH^*(\Lambda V; d)$$

Introduction

Our main purpose in this paper is to investigate the case of coformal spaces, those verifying

$$dV \subset \Lambda^2 V$$

(i.e., the diferential is purely quadratic). We will also propose some research directions to resolve completely the coformal case by induction on the nilpotency degree of the associated homotopy Lie algebra. We will test these ideas on some informative examples.

Proposition A. The Hilali conjecture holds for any coformal space X whose rational homotopy Lie algebra L is of nilpotency 1 or 2.

Proposition B. The toral rank conjecture holds for any manifold of dimension less than 16, and whose rational toral rank is equal to 4.

Hilali conjecture for coformal space

Through this section, $L = \pi_*(\Omega X) \otimes \mathbb{Q} \simeq \pi_{*+1}(X) \otimes \mathbb{Q}$ denotes the rational homotopy Lie algebra of X. Using the Cartan-Eilenberg-Chevalley construction, many connections between L and the related Sullivan model are established, here below some useful ones quoted from $[FHT01]$:

Definition

Soit (L, ∂) une algèbre de LIE différentielle graduée

$$C^*(L, \partial) = (\Lambda \sharp SL \ , \ d = d_1 + d_2)$$

$$< d_1 z; sx > \ = (-1)^{|z|} < z; s\partial x > \quad \forall z \in \sharp sL, x \in L$$

$$< d_2 z; sx_1 \Lambda sx_2 >= -(-1)^{|x_2|} < z; s\,[x_1, x_2] > \forall z \in \sharp sL, x_i \in L$$