Orbit Configuration Spaces

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Introduction

We propose In this talk the introduction of Orbit Configuration Spaces together with some interesting properties, notably the orbit configuration space corresponding to the natural action of a finite cyclic groups, historically, the study of Orbit Configuration Spaces has been introduced by M.Xicotencatl in his thesis in 1997 [], we investigate some problems related to this kind of spaces, among these problems, the generalization to any finite group and the crucial question that is Hilali Conjecture of Orbit Configuration Spaces.

Definition and Basic Properties

Let M be an $n-$dimensional connected manifold and G be a finite group and let us assume that G acts freely on M. Let Gx denote the orbit of an element x of M. under the action of G. We define the *Orbit Configuration Spaces* by

$$F_G(M,k) := \{(x_1,...,x_k) \in M^k; Gx_i \cap Gx_j = \emptyset \text{ if } i \neq j\}$$

or equivalently $F_G(M,k) := \{(x_1,...,x_k) \in M^k; Gx_i = Gx_j \text{ if } i = j\}$
The main relation to ordinary Configuration Spaces is given by the following result :

Theorem

Let G act on M such that the canonical projection $M \to M/G$ is a principal $G-$bundle. Then G^k acts on $F_G(M,k)$ and $F_G(M,k)/G^k \approx F(M/G,k)$.

• **Remarks :**

1-In the case where G acts trivially on M, we have $F_G(M,k) = F(M,k)$.

2- In particular, there is a principal G^k-bundle :

$$G^k \to F_G(M,k) \to F(M/G,k)$$

For any natural number i, fix a finite subset $Q_i \subset M$ with cardinality $|Q_i| = i$. Then the spaces $F_G(M,k)$ satisfy the following :

Theorem

For $k \geq l$, the projection $p : F_G(M,k) \to F_G(M,l)$ onto the first l coordinates, is a locally trivial bundle, with fibre $F_G(M - Q_{|G|l}, k-l)$

An equivalent definition can be given in terms of ordinary configuration spaces. Let $f : M/G \to BG$ be the map which classifies the covering space $G \to M \to M/G$.

Theorem

The space $F_G(M,k)$ is homeomorphic to the total space of the pull-back of the principal fibration $G^k \to (EG)^k \to (BG)^k$ along the composition

$$F(M/G,k) \hookrightarrow (M/G)^k \xrightarrow{f^k} (BG)^k$$

Some examples of manifolds with free group-actions are :

1. $\mathbb{R}^n - \{0\}$ with a $\mathbb{Z}/2-$action given by the antipodal map.
2. $\mathbb{C}^n - \{0\}$ with a $\mathbb{Z}/p-$action given by multiplication by a primitive $p-$th root of unity ζ_p.
3. The actions in 1. and 2. restrict to free actions of $\mathbb{Z}/2$ and $\mathbb{Z}/p-$ on the spheres $\mathbb{S}^n$ and $\mathbb{S}^{2n+1}$ respectively.
4. For any manifold M, the symmetric group on k letters $\mathcal{G}_k$ acts freely on the configuration space $F(M, k)$.

We recall the following fact ; let $F \to E \to B$ be a fibration with a cross-section $\sigma : B \to E$. Then there is a homotopy equivalence : $\Omega B \times \Omega F \to \Omega E$. This fact is useful to prove this interesting result :

Theorem

If the fibration $(M - Q_{|G|(i-1)}) \to F_G(M,i) \to F_G(M,i-1)$ *has a cross-section for* $2 \leq i \leq k$*, then there is a homotopy equivalence :*

$$\Omega F_G(M,k) \simeq \prod_{i=0}^{k-1} \Omega(M - Q_{|G|i})$$

The case $M = \mathbb{S}^n$, G=Z$_2$

In this case $F_{\mathbb{Z}_2}(\mathbb{S}^n, k) = \{(x_1, ..., x_k) \in (\mathbb{S}^n)^k; x_i \neq \pm x_j\}$, Let $p : F_{\mathbb{Z}_2}(\mathbb{S}^n, k) \to \mathbb{S}^n$ be the projection onto the first coordinate. By Fadell-Neuwirth theorem, p is a fibration with fibre $F_{\mathbb{Z}_2}(\mathbb{S}^n - \{\pm e_{n+1}\}, k-1)$.
We get thereby a fibration $F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \to F_{\mathbb{Z}_2}(\mathbb{S}^n, k+1) \xrightarrow{p} \mathbb{S}^n$

Cohomology of $F_G(M, k)$

Notice that the G^k-action on $F_G(M, k)$ induces an action on cohomology, and we have

Theorem

Let G be a finite group acting freely on a manifold M and let R a ring where $|G|$ is a unit. Then there is an isomorphism of algebras :

$$H^*(F(M/G, k); R) \cong H^*(F_G(M, k); R)^{G^k}$$

where $(-)^{G^k}$ denotes the module of invariants.

The key of the proof is the fact that the spectral sequence for a covering $E_2^{p,q} = H^p(G^k; H^q(F_G(M, k); R))$ converges to $H^*F(M/G, k)$

The cohomology of the total space split as the tensor product of the cohomology of the base and the cohomology of the fibre, by induction we get

$$H^* F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \cong \bigotimes_i^k H^*(\vee_{2i-1}\mathbb{S}^{n-1}).$$

And the rational cohomology of $F_{\mathbb{Z}_2}(\mathbb{S}^n, k)$ where n is odd is given by

$$H^*(F_{\mathbb{Z}_2}(\mathbb{S}^n, k); \mathbb{Q}) = H^*(\mathbb{S}^n; \mathbb{Q}) \otimes \bigotimes_i^{k-1} H^*(\vee_{2i-1}\mathbb{S}^{n-1}; \mathbb{Q}).$$

From the previous theorem, the isomorphism above leads to the following inequality :

$$\dim H^*(F(M/G, k); \mathbb{Q}) \leq \dim H^*(F_G(M, k); \mathbb{Q})$$

In [], Wu and Al. proved the following result by using combinatorial techniques ;

Theorem

If M is a smooth closed manifold of dimension m with action of $\mathbb{Z}_2^m$, then

$$\chi(F_{\mathbb{Z}_2^m}(M,k)) = \chi(F(M,k)).$$

Another interesting result from [] is :

Theorem

Under the same conditions as the theorem above, $F_{\mathbb{Z}_2}(M,k)$ has the same homotopy type as $k!2^{k-2}$ points, and $F_{\mathbb{S}^1}(M,k)$ has the same homotopy type as a disjoint union of $k!$ copies of the torus T^{k-2}.

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To Be Continued !

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