## Definitions & Motivation:

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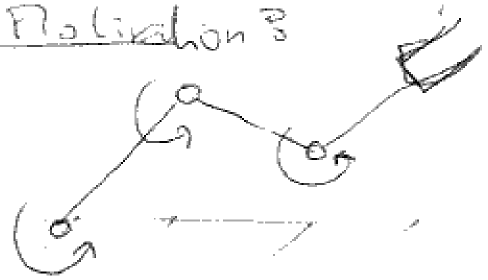


$X = S^1 \times S^1 \times S^1 \times S^1 = T^4$

The configuration space of a robot is a top space $X$ such that:

points of $X \longleftrightarrow$ physical states of robot

cts paths

$\gamma : [0,1] \to X \longleftrightarrow$ Motions of robots

$\cancel{X = S^1 \times S^1 \times S^1 = T^3}$

Let $PX$ be the space of paths $\boxed{1}$

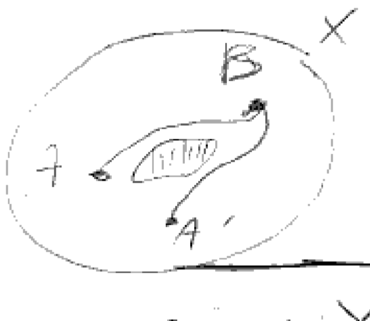
Free path fibration $\pi_X : PX \to X \times X$

$$\pi_X(\gamma) = (\gamma(0), \gamma(1))$$

A motion planning algorithm (m.p.a) in $X$ is a function $s : X \times X \to PX$ such that

$$\pi_X \circ s = Id_{X \times X}$$

Note: $s(A, B)$ is a path in $X$ from $A$ to $B$.




Prop. (Farber) $\exists$ a continuous m.p.a in $X$ $\Longleftrightarrow$ $X$ contractible $(X \simeq *)$

Définition A local m.p.a on $\mathcal{U} \subseteq X \times X$ is a fct

$s_{\mathcal{U}} : \mathcal{U} \longrightarrow PX$ such that

$\pi_X \circ s_{\mathcal{U}} = \mathrm{incl} : \mathcal{U} \longrightarrow X \times X$.

Définition (Farber):

The topological complexity of $X$

denoted $TC(X)$ is min $k \in \mathbb{N}$

such that $X \times X = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_k$

$\mathcal{U}_i \subseteq X \times X$ open,

$\exists s_i : \mathcal{U}_i \longrightarrow PX$ cts local m.p.a

$TC(X)$ homotopy invariant of pratical signifiance when designing m.p.a's.

## 2. Bounds

Dim/connectivity

If $\pi_i(X)$ trivial for $i \leq r$, then

$$TC(X) \leq \frac{2 \dim X}{r} + 1$$

If $r = 1$ then $TC(X) \leq 2 \dim X + 1$

Farber Svarc

A.S. Schwarz "The genus of a fiberspace"

(Translation AMS)

<u>L.S. Category</u> (1930's)

<u>Déf.</u> $\mathrm{Cat}(X)$ is min $k \in \mathbb{N}$ t.q.

$$X = U_1 \cup \dots \cup U_k$$

each $U_i \hookrightarrow X$ open

null-homotopic

<u>E.g.</u> $\mathrm{Cat}(S^n) = 2 \quad (n > 0)$

we have $\mathrm{cat}(X) \leq TC(X) \leq \mathrm{Cat}(X \times X)$

$$\leq 2\,\mathrm{Cat}(X)$$

<u>Cohomology</u>

$$H^*(X) = H^*(X; k) \text{ graded } k\text{-alg}$$

$$= \bigoplus_{i \geq 0} H^i(X, k)$$

$\boxed{2}$

$$H^i \times H^j \longrightarrow H^{i+j}$$

$$H^*(X \times X) = H^*(X) \otimes_k H^*(X)$$

(if $k$ field, $X$ CW-complex of finite type)

Cup-product

$$H^*(X) \otimes H^*(X) \xrightarrow{\ \cup\ } H^*(X)$$ is an alg homomorphism

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}\, ac \otimes bd$$

$Ker(\cup)$ is a ideal of zero-divisors

we have $TC(X) > \text{cup-lenght}(Ker(\cup))$

If $u_1, \dots, u_k \in Ker(\cup)$ and $u_1 u_2 \dots u_k \neq 0$

$TC(X) > k$

# 3. Examples

(a) ~~Sphere~~ Spheres The n-sphere $S^n$ $n>0$ has

$\pi_i(S^n)$ trivial for $i<n$,

so $2 \leq TC(S^n) \leq \frac{2n}{n} + 1 = 3$

$$H^*(S^n) = \frac{k[x]}{x^2}, \quad x \in H^n(S^n)$$

$$(1\otimes x - x\otimes 1)^2 = (1\otimes x - x \otimes 1)(1\otimes x - x\otimes 1)$$

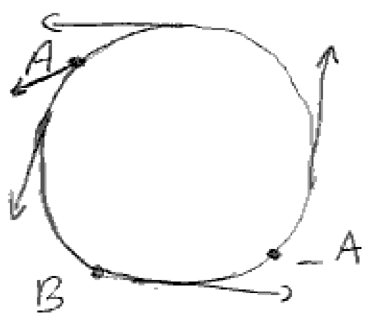
$$= -x\otimes x - (-1)^{n^2}\, x\otimes x$$

$$= \begin{cases} 0 & n \text{ odd} \\ -2x\otimes x & n \text{ even} \end{cases}$$

<u>If n even</u> $TC(S^n) = 3$

<u>If n odd</u>,

fix a nowhere zero vector field $v$ on $S^n$




$U_1 = \{(A,B) / B \neq -A\}$

$S_1 =$ Shortest geodesic A to B

$U_2 = \{(A,B) / A \neq B\}$

$S_2 =$ geodesic in direction $v(A)$

## (b) Orientables surfaces

$M_g$ = orientable surface genus $g$

g holes

**Exercise** $TC(M_g) = \begin{cases} 3 & g=0,1 \\ 5 & g>1 \end{cases}$

**Hint** show $TC(G) = Cat(G)$, $G$ Lie group

## 4. Open problems

(a) $N_g$ = non-orientable surface, genus $g$

($N_1 = \mathbb{RP}^2$
$N_2$ = Klein bottle  )

$$TC(N_g) = \begin{cases} 4 & , g=1 \\ 4 \text{ or } 5 & , g=2 \end{cases}$$

$\boxed{3}$

(b) Symmetric TC

$PX$, $X \times X$ admit involutions (action of $\mathbb{Z}/2$)

$\gamma \to \bar{\gamma}$ (reverse path)

$(A, B) \to (B, A)$, $\pi_X : PX \to X \times X$ is equivariant.

Basabe-González-Rudyak-Tamaki

$TC^{\Sigma}(X) = \min k \in \mathbb{N}$ tq $X \times X = U_1 \cup \dots \cup U_k$

each $U_i \subseteq X \times X$ open and $\mathbb{Z}/2$ invariant

$\left((A,B)\in \mathcal{U}_i \Rightarrow (B,A)\in \mathcal{U}_i\right)$, $\exists s_i : \mathcal{U}_i \longrightarrow PX$

cts, equivariant local m.p.a

<u>Question</u> $TC^{\Sigma}(S^n) = ?$

(C) $TC_0(X) = TC(X_{\mathbb{Q}})$

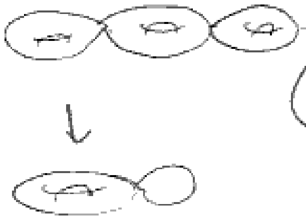
can be described in terms of Sullivan's minimal models (Jessup - Murillo - Parent - Carrasquel)

<u>Rational TC Gener. Conj</u>:

$$TC_0(X \times S^n) = TC_0(X) + TC_0(S^n)$$

$TC(\tilde{X}) \le TC(X)$

in general



(Dranishnikov)