Spaces of topological complexity two

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1. Definitions

2. About spaces of $TC_n(X) = n$

We recall that $cat(X)$ is the smallest n for which there is an open covering $\{U_0, ..., U_n\}$ by $(n+1)$ open sets, each of which is contractible in X.
The sectional category of a fibration $p : E \longrightarrow B$, denoted by $secat(p)$, is the smallest number n for which there is an open covering $\{U_0, ..., U_n\}$ of B by $(n+1)$ open sets, for each of which there is a local section $s_i : U_i \longrightarrow E$ of p, so that $p \circ s_i = j_i : U_i \longrightarrow B$, where j_i denotes the inclusion.

Let PX denote the space of (free) paths on a space X. There is a fibration $P_2 : PX \longrightarrow X \times X$, which evaluates a path at initial and final point : for $\alpha \in PX$, we have $P_2(\alpha) = (\alpha(0), \alpha(1))$. This is a fibrational substitute for the diagonal map $\Delta : X \longrightarrow X \times X$. We define the topological complexity $TC(X)$ of X to be the sectional category $secat(P_2)$ of this fibration. That is, $TC(X)$ is the smallest number n for which there is an open cover $\{U_0, ..., U_n\}$ of $X \times X$ by $(n+1)$ open sets, for each of which there is a local section $s_i : U_i \longrightarrow PX$ of P_2, i.e., for which $P_2 \circ s_i = j_i : U_i \longrightarrow X \times X$, where j_i denotes the inclusion.

More generally, let $n \geq 2$ and consider the fibration

$$P_n : PX \longrightarrow X \times X \times \cdots \times X = X^n$$

defined by dividing the unit interval $I = [0,1]$ into $(n-1)$ subintervals of equal length, with n subdivision points $t_0 = 0, t_1 = 1/(n-1), \ldots, t_{n-1} = 1$ (thus $(n-2)$ subdivision points interior to the interval), and then evaluating at each of the n subdivision points, thus :

$$P_n(\alpha) = (\alpha(0), \alpha(1), \ldots, \alpha(t_{n-2}), \alpha(1))$$

for $\alpha \in PX$. This is a fibrational substitute for the n-fold diagonal $\Delta_n : X \longrightarrow X^n$. Then the higher topological complexity $TC_n(X)$ is defined as $secat(P_n)$.

$f : X \longrightarrow Y$ is called a weak homotopy equivalence if it induces isomorphisms $\pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ for all $n \geq 0$

whitehead theorem

A weak homotopy equivalent between C.W. complexes is a homotopy equivalence.

Proposition

$TC_n(S^k) = n - 1$ for k odd and $TC_n(S^k) = n$ for k even.

Questions

if X is a C.W. complexe simply connected integral homology sphere and $TC_n(X) = n$. Would we have X is of the homotopy equivalent to some sphere S^k for k even. ? ?

Questions

In particular, it's true if X is a weak homotopy equivalent to S^k, for k even, by whitehead theorem.

Questions

Using the inequality of theorem 3.1. in "On TC of twested products" $TC(X) \leq TC(B) + TC^*_G(F)$, Giving a space X which is total space of a fibration with basis odd dimentional sphere, we can try to determine such fibrations where $TC^*_G(F) = 0$ so that we will have exemples of spaces such that $TC(X) = 2$ with condition that X is non contractible and not homotopy equivalent to an odd dimentional sphere.